Polynumber field theory (презентация доклада на семинаре с участием Р. Пенроуза 03.04.2013)

С.С. Кокарев[1], НИИ ГСГФ (Фрязино) — РНОЦ "Логос" (Ярославль)

Polynumber field theory

S.S. Kokarev, RI HSGP (Fryazino) — RSEC "Logos" (Yaroslavl)

[1]e-mail: logos-center@mail.ru

Polynumber field theory

(double numbers variations on complex motives)

S. S. Kokarev
RI HSGP (Fryazino) — RSEC "Logos"
(Yaroslavl)

1. Introduction: double numbers are "neighbors" of complex ones!

$$\mathbb{A}_2 \ni A = a + Ib, \quad a, b \in \mathbb{R}.$$

$\mathbb{A}_2$ is defined by the pair $p, q \in \mathbb{R}$:

$$I^2 = p + Iq \Rightarrow (I - q/2)^2 = p + q^2/4$$

$$\begin{cases} \text{sign}(p + q^2/4) = -1, \to \mathbb{A}_2 \sim \mathbb{C}; \\ \text{sign}(p + q^2/4) = +1, \to \mathbb{A}_2 \sim H; \\ \text{sign}(p + q^2/4) = 0, \quad \to \mathbb{A}_2 \sim \Omega. \end{cases} \quad (1)$$

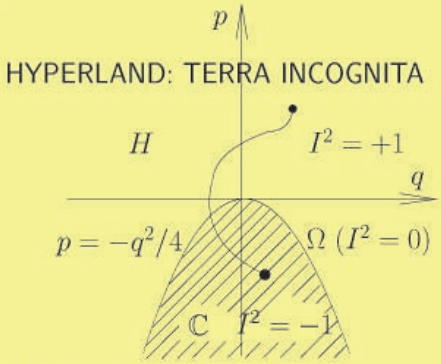


Fig. 1. Two-dimensional commutative-associative algebras $\mathbb{A}_2$ over $\mathbb{R}$.

2

2. The algebra $\mathbb{C}$ and its applications

1. **Algebra of complex numbers.**

$$\mathbb{C} = \{z = x + iy \,|\, i^2 = -1,\ x, y \in \mathbb{R}\} \quad (2)$$

— algebraically-closed number field (main theorem of algebra).

2. **Euclidean geometry of $\mathbb{C}$.**

$$\eta \equiv \mathrm{Re}(dz_1 \otimes d\bar{z}_2) = dx_1 \otimes dx_2 + dy_1 \otimes dy_2$$

— Euclidean metric.

3. **Linear fractional transformations.**

$$\mathrm{DL}(\mathbb{C}) \sim \mathrm{SL}(2, \mathbb{C}) \quad (3)$$

Invariance of circles and of the anharmonic relation:

$$\{z_1, z_2, z_3, z_4\} = \frac{z_1 - z_2}{z_1 - z_4} \cdot \frac{z_3 - z_4}{z_3 - z_2} \quad (4)$$

4. **Stereographic projection.**

$$\Pi : S^2 \leftrightarrows \mathbb{C} \cup \{\infty\}. \quad (5)$$

Conformal property. Invariance of circles.

3

5. **Spinors.** $\xi, \zeta \in \mathfrak{S} : \langle \xi, \zeta \rangle \equiv *(\xi \wedge \zeta)$.

$$\begin{array}{ccc} \mathsf{SL}(2,\mathbb{C}) & \longrightarrow & \mathsf{SU}(2) \\ 2{:}1 \Big\downarrow & & \Big\downarrow 2{:}1 \\ \mathsf{SO}(1,3) & \longrightarrow & \mathsf{SO}(3) \end{array} \qquad (6)$$

Transition to vectors of $\mathcal{M}_4$: $V \sim \mathfrak{S} \otimes \bar{\mathfrak{S}}$.

6. **Mappings** $\mathcal{H}\mathsf{ol}(\mathbb{C})$.

General differentiability:

$$dF = A(z,\bar{z})dz + B(z,\bar{z})d\bar{z}. \qquad (7)$$

$\mathbb{C}$-differentiability:

$$B = 0 \Leftrightarrow \frac{\partial F}{\partial \bar{z}} = \frac{\partial U}{\partial x} - \frac{\partial V}{\partial y} + i\left(\frac{\partial V}{\partial x} + \frac{\partial U}{\partial y}\right) = 0$$

— C-R conditions for $F = U + iV$.

Harmonicity of U and V. Infinite differentiability, analyticity, Taylor's, Laurent and Puiseux series.

7. **Topological aspects of** $\mathcal{H}\mathsf{ol}(\mathbb{C})$

Cauchy theorem and formula:

$$\oint_{\mathcal{C}} F(z)\,dz = 0; \;\; F(z_0) = \frac{1}{2\pi i}\oint_{\mathcal{C}} \frac{F(z)}{z - z_0}.$$

4

Analytic continuation, multi-valued functions, Riemann surfaces and residue theory.

8. **Geometrical aspect of $\mathcal{H}$ol$(\mathbb{C})$.**

Conformal property:

$$\eta \mapsto \eta' = |F'(z)|^2 \eta, \quad F'(z) \neq 0. \tag{8}$$

Orthogonality of levels lines:

$$\nabla U \cdot \nabla V = 0 \tag{9}$$

Passive (change of coordinates) and active (deformation) interpretation. Flatness criteria for conformally flat 2-dimensional metric:

$$\eta_\phi = e^{2\phi} \eta \tag{10}$$

— existence of $F(z)$, such that $\phi = \ln |F|$.

9. **Physical aspect of $\mathcal{H}$ol$(\mathbb{C})$.**

Electro- and magnetostatics, hydrodynamics, elasticity theory, scattering theory etc.

C-R conditions $\Rightarrow$ potentiality and solenoidality of physical fields.

$F(z) = \varphi + i\chi$ — complex potential, φ — potential function, χ — force (flow) function.

5

Direct method:

Boundary conditions $\Rightarrow$ straightening of boundaries of conductors or flow profiles by means of $F(z) \overset{\text{uniqueness th.}}{\Rightarrow}$ solution!

Inverse method:

Any holomorphic function $F(z) \Rightarrow$ "free" solution to some flat problem!

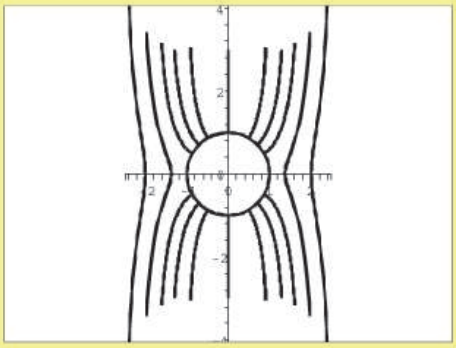


Fig. 2. Complex potential $F(z) = E_0(z + R^2 z^{-1})$ (conductive neutral cylinder in homogeneous external field)

Dual transformation:

$$F \to iF \Rightarrow B = iE, \tag{11}$$

6

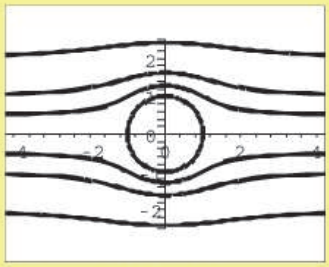


Fig. 3. Dual interpretation (diamagnetic cylinder in homogeneous external field).

10. **Fractal aspect of $\mathcal{H}$ol($\mathbb{C}$).**

Iteration process: $z_{k+1} = f(z_k)$.

For $f(z) = z^2 + c$ we obtain **the Julia set** and **the Mandelbrojt fractal.**

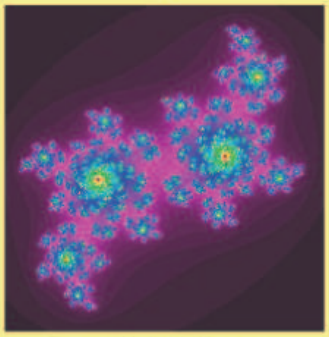


Fig. 4. The Julia set.

7

3. Algebra H of double numbers.

Generators of H: $\{1, j\}$;

$$H = \{h = t + jx \,|\, j^2 = +1,\ t, x \in \mathbb{R}\}. \quad (12)$$

Real, imaginary parts and complex conjugation:

$$\mathrm{Re}h = t; \quad \mathrm{Im}h = x; \quad \bar{h} = t - jx. \quad (13)$$

Metric form:

$$\epsilon \equiv \mathrm{Re}(dh \otimes d\bar{h}) = dt \otimes dt - dx \otimes dx$$

— 2-dimensional pseudo-Euclidean metric.

(Pseudo)norm and module of h:

$$\|h\|^2 \equiv h\bar{h},\ (\|hg\|^2 = \|h\|^2\|g\|^2),\ |h| \equiv \sqrt{|\|h\|^2|}.$$

Degenerate elements (zero divisors):

$$H^\circ \equiv \{h = a(1 \pm j)\}; \quad \|h\|^2 = 0. \quad (14)$$

8

Inversion of non-degenerate elements:

$$h \mapsto h^{-1} \equiv \frac{\bar{h}}{\bar{h}h}. \qquad (15)$$

Polar coordinates on H:

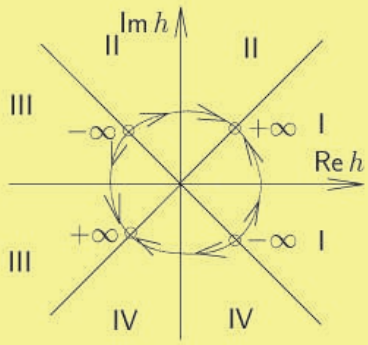


Fig. 5. $\psi = \mathbb{R} \sqcup -\mathbb{R} \sqcup \mathbb{R} \sqcup -\mathbb{R}$

$$h = t + jx = I\varrho(\cosh\psi + j\sinh\psi), \qquad (16)$$

$$\begin{aligned}
\mathsf{I}: \ & I = 1, & \varrho &= \sqrt{t^2 - x^2}, & \psi &= \mathsf{Arcth}(x/t);\\
\mathsf{II}: \ & I = j, & \varrho &= \sqrt{x^2 - t^2}, & \psi &= \mathsf{Arcth}(t/x);\\
\mathsf{III}: \ & I = -1, & \varrho &= \sqrt{t^2 - x^2}, & \psi &= \mathsf{Acrth}(x/t);\\
\mathsf{IV}: \ & I = -j, & \varrho &= \sqrt{x^2 - t^2}, & \psi &= \mathsf{Arcth}(t/x).
\end{aligned}$$

ϱ — module, ψ — argument.

The directions $\psi = \pm\infty \Leftrightarrow H^\circ$ (polar coordinates do not work!).

9

The cone of h_0:

$$\mathrm{Con}(h_0) = \{h \in H \,|\, h - h_0 \in H^{\circ}\} \qquad (17)$$

The geometry of H is the pseudo-Euclidean geometry of 2-dimensional Minkowski ST!

Isometry group — 3-dimensional Poincare group + reflections:

$h \mapsto h+a;\ h \to l_{\psi}\cdot h,\ l_{\psi} \equiv \cosh\psi + j\sinh\psi.$

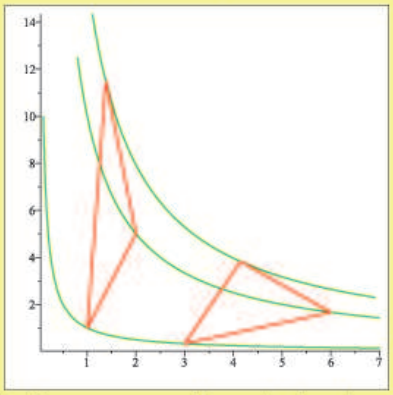


Fig. 6. Congruent equilateral triangles on H.

10

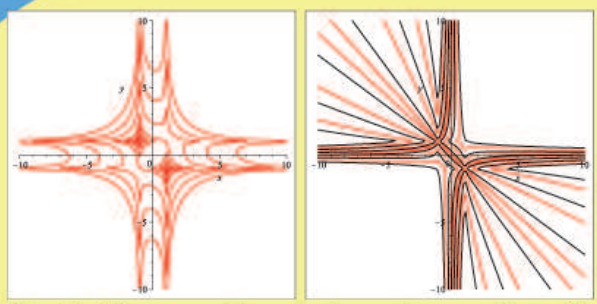


Fig. 7. Ellipses and hyperbolas $\varrho_1 \pm \varrho_2 = C$ on H.

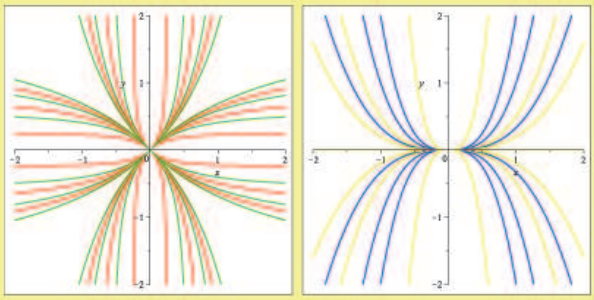


Fig. 8. Spirals ($\varrho = C\psi$ and $\varrho = Ce^{\psi}$) on H.

4. Elementary functions on H

What is $F(t + jh)$?

1. Real law for "superposition of arguments" :

$$f(x + y) = \Phi(f_1(x), f_2(y));$$

2. For $f(x) = (Sf)(x) + (Af)(x)$:

$$(Sf)(jx) = (Sf)(x);\ (Af)(jx) = j(Af)(x)$$

— interaction of symmetric and antisymmetric components of f with j.

These rules are sufficient for the construction of analytical extension of all elementary functions from $\mathbb{R}$ to H!

12

1. e^h and $\ln h$

$$h \mapsto e^h = e^t e^{jx} = e^t(\cosh x + j \sinh x),$$

$$H_I \ni h \to \ln h = \ln \varrho + j\psi.$$

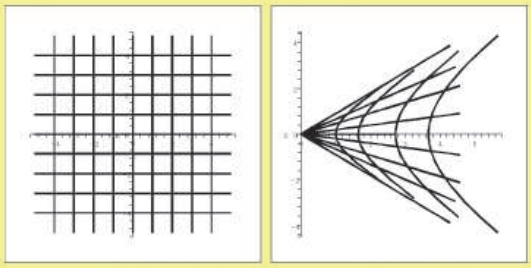


Fig. 9. The global structure of e^h and $\ln h$.

2. h^n, $n \in \mathbb{Z}$. $\quad h = I\varrho e^{j\psi} \overset{h^n}{\mapsto} I^n \varrho^n e^{jn\psi}$.

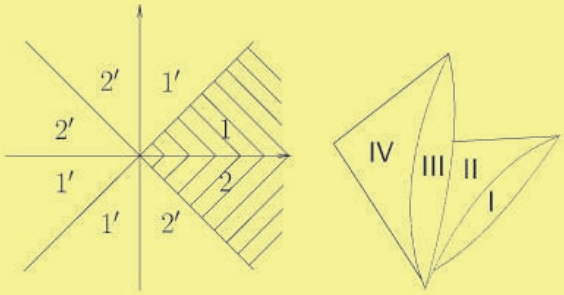


Fig. 10. The mapping $h \mapsto h^{2k}$

13

The number $N(P)$ of roots of polynomial $P_n(h)$ in general case satisfies the inequality:

$$0 \leq N(P_n) \leq n^2.$$

3. Trigonometric, hyperbolic functions and their inversion

$$\sin h = \sin t \cos x + j \sin x \cos t. \qquad (18)$$

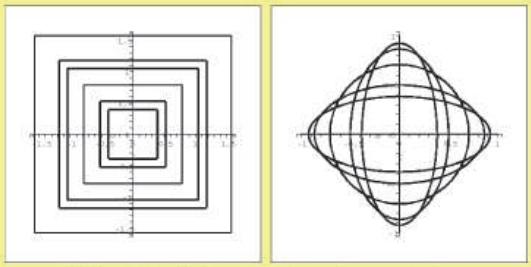


Fig. 11. The mapping $h \mapsto \sin h$.

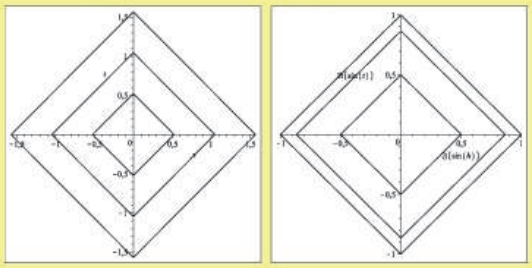


Fig. 12. Fundamental squares for $\sin h$.

14

$$\arcsin\, h = \frac{1}{2}[\arcsin((t+x)\sqrt{1-(t-x)^2}+ (t-x)\sqrt{1-(t+x)^2})+ j\arcsin((t+x)\sqrt{1-(t-x)^2}- (t-x)\sqrt{1-(t+x)^2})] \quad (19)$$

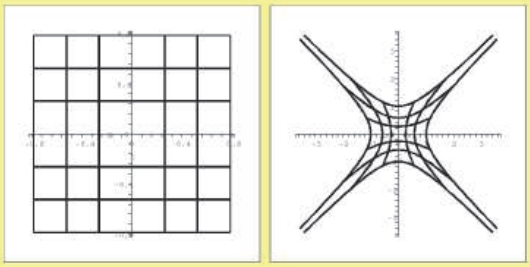


Fig. 13. The mapping
$h \mapsto \tan h = (\sin 2t + j \sin 2x)/(\cos 2t + \cos 2x)$.

$$\arctan\, h = \frac{1}{2}\left\{\arctan\left[\frac{2t}{1-t^2+x^2}\right] + j\arctan\left[\frac{2x}{1+t^2-x^2}\right]\right\}. \quad (20)$$

15

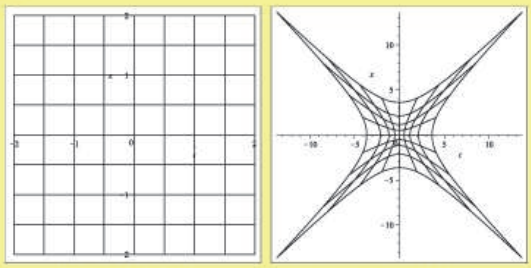


Fig. 14. The mapping $h \mapsto \sinh h$.

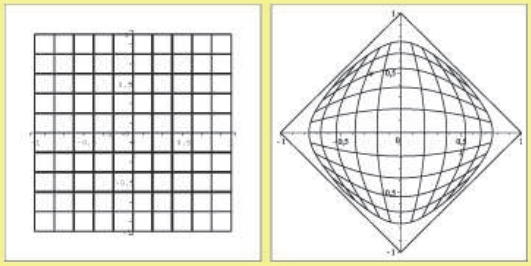


Fig. 15. The mapping $h \mapsto \tanh h$.

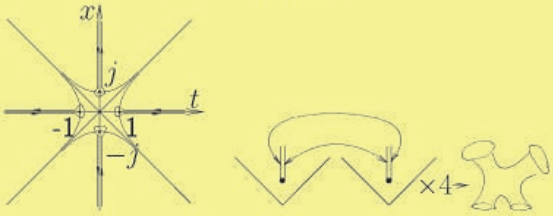


Fig. 16. Mapping $h \mapsto (h + h^{-1})/2$ — the hyperbolic Zhukowski function.

16

5. Isotropic basis

$$e_1 = \frac{1+j}{2}; \quad e_2 = \frac{1-j}{2},$$

$$t = \frac{h_1 + h_2}{2}, \; x = \frac{h_1 - h_2}{2}.$$

Table of composition and conjugation:

$$e_1^2 = e_1; \; e_2^2 = e_2; \; e_1 e_2 = 0; \; \bar{e}_1 = e_2. \quad (21)$$

Product, unit and (pseudo)norm:

$$h \cdot g = h_1 g_1 e_1 + h_2 g_2 e_2; \; 1 = e_1 + e_2; \; \|h\|^2 = h_1 h_2.$$

Integer power:

$$h^n = h_1^n e_1 + h_2^n e_2. \quad (22)$$

Power series:

$$\sum_k c_k h^k = \left(\sum_k c_k^1 h_1^k \right) e_1 + \left(\sum_k c_k^2 h_2^k \right) e_2.$$

17

6. Analytic extensions

$J \subseteq \mathbb{R}$, $D \subseteq \mathbb{R}^2$.

St. 1

$f \in C^\omega(J) \;\Leftrightarrow \hat{f} \in C^\omega(J \times J) : H \to H$

$\hat{f}(h) = f(h_1)e_1 + f(h_2)e_2.$

St. 2

$f \in C^\omega(J_1) \times C^\omega(J_2) \;\Leftrightarrow \hat{f} \in C^\omega(J_1 \times J_2)$:

$f_1(h_1)e_1 + f_2(h_2)e_2 = \sum c_k h^k \equiv \hat{f}(h).$

St. 3

$f \in C^\omega(D_1) \times C^\omega(D_2) \Leftrightarrow \hat{f} \in C^\omega(D_1 \cap D_2)$:

$\hat{f}(h, \bar{h}) = f_1(h_1, h_2)e_1 + f_2(h_1, h_2)e_2,$

$c_{km} = c^1_{km} e_1 + c^2_{km} e_2$

St. 4

$f \in C^\omega(\gamma) \;\Rightarrow \hat{f} \in C^\omega(\pi_1(\gamma) \times \pi_2(\gamma))$:

$\hat{f}(h) = f \circ \gamma^{-1}(h).$

(Reinhardt domains — rectangles!)

18

7. Compactification of H

$$H \to H^{\square} = H \cup L_{\infty} \cup \Omega_{\infty}, \qquad (23)$$

$$L_{\infty} = \{(-\infty, a)\} \cup \{(+\infty, a)\} \cup \{(a, -\infty)\} \cup \{(a, +\infty)\} \qquad (24)$$

$$\Omega_{\infty} = \{(+\infty, +\infty)\} \cup \{(-\infty, +\infty)\} \cup \{(-\infty, -\infty)\} \cup \{(+\infty, -\infty)\}. \qquad (25)$$

$$H^{\square} \to H^{\circledcirc} = H_0^{\square} \cup S_{1\infty} \cup S_{2\infty} \cup O_{\infty}. \qquad (26)$$

The compactifying mapping: $h \mapsto \sin(\pi \tanh h)$.

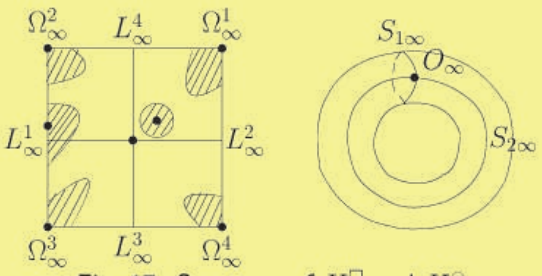


Fig. 17. Structure of $H^{\square}$ and $H^{\circledcirc}$.

19

8. Linear fractional transformations over H.

$$\mathsf{DL}(H) \ni D_M : H^{\circledcirc} \to H^{\circledcirc}, \quad M = \begin{pmatrix} a & b \\ c & d \end{pmatrix},$$

$$D_M(h) = \frac{ah+b}{ch+d} = \frac{a_1h_1+b_1}{c_1h_1+d_1}e_1 + \frac{a_2h_2+b_2}{c_2h_2+d_2}e_2$$

Properties of DL(H).

1. DL(H) $\sim$ SL(2,H).
2. $D_M \in C^{\omega}(H^{\circledcirc} \setminus \mathsf{Con}(-d/c))$;
3. $D_M(-d/c) = O_\infty$, $D_M(\mathsf{Con}(-d/c)) = \mathsf{Con}(O_\infty) = S_{1\infty} \cup S_{2\infty}$.;
4. $D_M(O_\infty) = a/c$, $D_M(\mathsf{Con}(O_\infty)) = \mathsf{Con}(a/c)$.
5. $h \overset{S_H}{\sim} h' \Leftrightarrow D_M(h) \overset{D_M(S_H)}{\sim} D_M(h')$ — $\mathsf{DL}(H)$ preserve conjugated points (circle property).
6. $\{g, h, w, u\}$ = inv, on the connected circle components it is real!

20

Automorphisms of the unit circle:

$$h' = \pm e^{j\psi}\frac{h-a}{1-\bar{a}h}. \tag{27}$$

Automorphisms of the half-plane $\operatorname{Im} h \geq 0$:

$$M \in \mathsf{SL}(2,\mathbb{R}). \tag{28}$$

Example:

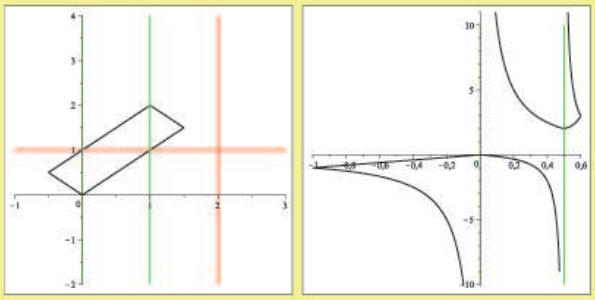


Fig. 18. $h \mapsto (h-a)/(h-\bar{a})$, $a = e_1 + 2e_2$.

21

The model of Lobachevski geometry: straight lines — hyperbolic circles in the upper half-plane (a pair of points defines a straight line, the real axis is Absolute).

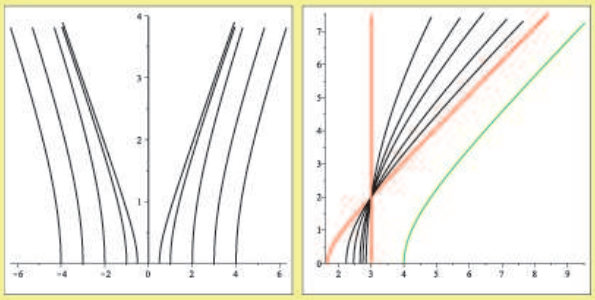


Fig. 19. Double number realization of Lobachevski geometry.

$$\rho(h,g) \equiv |\ln\{h,g,O_\infty,a_{(h,g)}\}| = \left|\ln\frac{1-\sinh\psi_h-\cosh\psi_h}{1-\sinh\psi_g-\cosh\psi_g}\right|.$$

$$\rho(h,O_\infty)=\rho(h,A)=\infty, \quad A\in \mathrm{Re}h.$$

22

9. Hyperbolic spinors $\mathfrak{S}(H)$.

$\mathfrak{S}(H)$ — 2-dimensional H-module with the metric:

$$\langle \xi, \zeta \rangle \equiv *(\xi \wedge \zeta), \quad \xi, \zeta \in \mathfrak{S}(H).$$

The chain of inclusions of symmetry groups:

$$\mathsf{SL}(2, H) \supset \mathsf{SU}(2, H) \supset \mathsf{SO}_2(\mathbb{R}), \quad (29)$$

where $\mathsf{SL}(2, H)$ — 6-parametric group ($\sim \mathsf{SL}(2, \mathbb{R}) \times \mathsf{SL}(2, \mathbb{R})$)

$$M = A_1 e_1 + A_2 e_2, \quad \det A_1 = \det A_2 = 1;$$

$\mathsf{SU}(2, H)$ — 3-parametric group of H-unitary transformations ($\sim \mathsf{SL}(2, \mathbb{R})$):

$$M = A e_1 + (A^{\mathsf{T}})^{-1} e_2, \quad \det A = 1;$$

$\mathsf{SO}_2(\mathbb{R})$ — 1-parametric real group of rotations:

$$M = R(e_1 + e_2), \quad R R^{\mathsf{T}} = 1.$$

23

In the vector space

$$\mathfrak{E}(H) \equiv S(\mathfrak{S}(H) \otimes \bar{\mathfrak{S}}(H))$$

we have for some $K \in \mathfrak{E}(H)$:

$$K = \begin{pmatrix} T+X & Y+jU \\ Y-jU & T-X \end{pmatrix} \mapsto K' = M \cdot K \cdot M^{\dagger}.$$

<u>Invariants</u>:

$$\det K = T^2 + U^2 - X^2 - Y^2, \; M \in \mathsf{SL}(2,H),$$

$$\mathsf{Trace}\, K = 2T, \quad M \in \mathsf{SU}(2,H),$$

$$\mathsf{Im}\, K = U\langle\ ,\ \rangle, \quad M \in \mathsf{SO}_2(\mathbb{R}).$$

<u>Diagram of homomorphims</u>:

$$\begin{array}{ccccc} \mathsf{SL}(2,H) & \to & \mathsf{SU}(2,H) & \to & \mathsf{SO}_2(\mathbb{R}) \\ 4{:}1 \downarrow & & \downarrow 4{:}1 & & \downarrow 4{:}1 \\ \mathsf{SO}(2,2) & \to & \mathsf{SO}(1,2) & \to & \mathsf{SO}_2(\mathbb{R}) \end{array} \qquad (30)$$

The spinor structure "senses" the parity of elements of the fundamental group ($\pi_1(\mathsf{SO}(2,2)) \sim Z \oplus Z$)!

24

10. h-holomorphic functions $\mathcal{H}$ol(H).

Common differentiability:

$$\Delta F = A(h,\bar{h})\,\Delta h + B(h,\bar{h})\,\Delta\bar{h} + o(\Delta h) + o(\Delta\bar{h}).$$

In the isotropic basis:

$$\Delta F = (A_1\Delta h_1 + B_1\Delta h_2 + o(\Delta h_1) + o(\Delta h_2))e_1 + (A_2\Delta h_2 + B_2\Delta h_1 + o(\Delta h_1) + o(\Delta h_2))e_2.$$

Hyperbolic holomorphicity:

$$B = 0 \Rightarrow \Delta F = A(h)\Delta h + o(\Delta h) = (A_1(h_1)\Delta h_1 + o(\Delta h_1))e_1 + (A_2(h_2) + o(\Delta h_2))e_2.$$

So, we have the following bijection:

$$\mathcal{H}\text{ol}(H) \sim C^1(\mathbb{R}) \times C^1(\mathbb{R}).$$

25

For the definition of derivatives the following additional rule for the division of zero divisors is sufficient:

$$\frac{\lambda e_1 + 0 \cdot e_2}{\sigma e_1 + 0 \cdot e_2} \equiv \frac{\lambda}{\sigma} e_1 + 0 \cdot e_2. \qquad (31)$$

Formal operators:

$$\frac{\partial}{\partial h} = \frac{1}{2}\left(\frac{\partial}{\partial t} + j\frac{\partial}{\partial x}\right); \quad \frac{\partial}{\partial \bar{h}} = \frac{1}{2}\left(\frac{\partial}{\partial t} - j\frac{\partial}{\partial x}\right). \qquad (32)$$

The equivalent definition of holomorphic $F = U + jV$:

$$F_{,\bar{h}} = 0 \Rightarrow U_{,t} - V_{,x} + j(V_{,t} - U_{,x}) = 0$$
$$\Rightarrow U_{,t} = V_{,x}; \quad U_{,x} = V_{,t}. \qquad (33)$$

— hyperbolic C-R conditions.

$\mathcal{H}\mathrm{ol}(H) \neq C^{\omega}(H)$

Example:

$$\Theta(h) \cdot e^{-1/h^2} \equiv \theta(h_1) e^{-1/h_1^2} e_1 + \theta(h_2) e^{-1/h_2^2} e_2.$$

26

11. Conformal property.

Law of transformation for the metric:

$$\epsilon = dh \otimes dh \xrightarrow{F} \tilde{\epsilon} = \|F'\|^2 \epsilon \qquad (34)$$

Transformations for length and area:

$$\widetilde{dl} = |F'|\, dl; \quad \widetilde{(dh \wedge d\bar{h})} = \|F'\|^2 (dh \wedge d\bar{h}).$$

Holomorphic functions generate conformal mappings $H \to H$ (they preserve hyperbolic angles under the condition $F' \neq 0$.).

Consequence — invariance of isotropic directions!

27

12. h-harmonic functions

Consequence of holomorphicity of F:

$$\Box F_1 = \Box F_2 = 0, \quad \Box \equiv \frac{1}{4}\frac{\partial^2}{\partial h \partial \bar{h}}, \qquad (35)$$

— h-harmonicity condition (in any basis).

F_1 and F_2 are conjugate h-harmonic functions (they locally define each other up to a constant).

Example:

$$U = \phi^+ + \phi^- \Rightarrow V = \phi^+ - \phi^- + C$$

Physical-geometrical meaning of the C-R conditions: the vector field

$$\bar{F} = (U, -V)$$

is h-potential and h-solenoidal:

$$\text{rot}\,\bar{F} \equiv U_{,x} - V_{,t} = 0; \quad \text{div}\,\bar{F} \equiv U_{,t} - V_{,x} = 0.$$

The net provided by $U = \text{const}$, $V = \text{const}$ is pseudo-orthogonal!

28

13. The Cauchy theorem and formula

Integration along a path on H:

$$\int\limits_{\Gamma} F\,dh \equiv \left(\int\limits_{\pi_1(\Gamma)} F_1|_{\Gamma}\,d\xi_1 \right) e_1 + \left(\int\limits_{\pi_2(\Gamma)} F_2|_{\Gamma}\,d\xi_2 \right) e_2$$

Integration of functions with singular points requires regularization!

Isotropic regularization:

$$\oint\limits_{C} F\,dh = 0 \Rightarrow \oint\limits_{C} \frac{F(h)}{h-h_0}\,dh = 0 \qquad (36)$$

for every piece-wise continuous F (the projections $\pi_i(C)$ are 1-dimensional "round-trip" paths on $\mathbb{R}$.)

29

14. Hyperbolic fields of elementary sources

Point-like source:

$$F(h) = -q \ln h,$$

$$\mathcal{E} = -\frac{d\bar{F}}{d\bar{h}} = \frac{qh}{\|h\|^2} = q\left(\frac{t}{t^2 - x^2} + j\frac{x}{t^2 - x^2}\right).$$

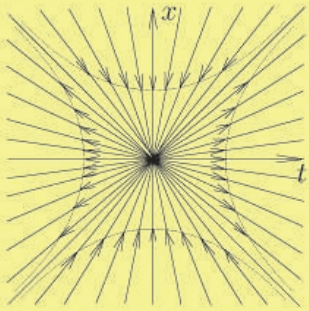


Fig. 20. Point-like hyperbolic source

30

Dual interpretation: $F(h) \to -jF(h)$.

Point-like vortex (source with imaginary charge):

$$\mathcal{B} = -j\frac{d\bar{F}}{d\bar{h}} = \frac{jq}{\bar{h}} = q\frac{x+jt}{t^2-x^2}. \quad (37)$$

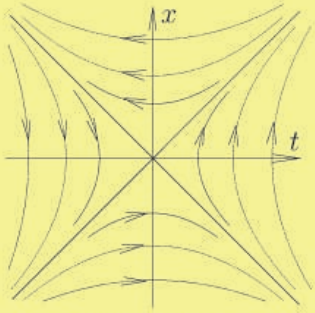


Fig. 21. Point-like vortex

31

Vortex-source (source with complex charge $Q = q - jm$):

$$F(z) = -Q \ln h = -q \ln \varrho + m\psi + j(m \ln \varrho - q\psi).$$

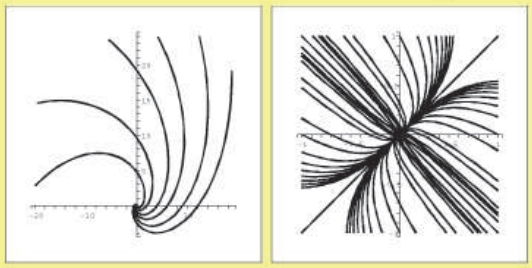


Fig. 22. Euclidean and hyperbolic vortex-source

Multipole source:

$$\mathcal{E}_n = Q^{(n)} \frac{d^n \ln \bar{h}}{d\bar{h}^n} = I_Q \frac{|Q^{(n)}| e^{-j(n\psi - \delta_Q)}}{\varrho^n},$$

$$\delta_Q = \text{Argh}\, Q^{(n)}.$$

Equation of force lines:

$$\varrho = C \sqrt[n-1]{|\sinh((n-1)\psi - \delta_n)|}, \quad (n > 1).$$

32

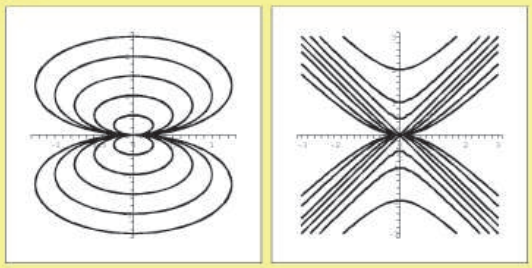


Fig. 23. $n = 2$: Euclidean and hyperbolic dipoles

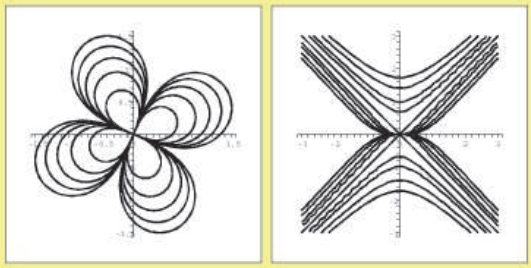


Fig. 24. $n = 3$: Euclidean and hyperbolic quadruples

15. 2-dimensional SR

The algebra H — space of events ($\sim \mathcal{M}_2$.)

The coalgebra H_* — generalized reference frames over $\mathcal{M}_2$.

The basis $\{1_*, j_*\}$ in H_*:

$1_*(1) = 1;\ 1_*(j) = 0;\ j_*(1) = 0;\ j_*(j) = 1.$

Inertial RF in SR:

$S_* = S^1_+.$

Subluminal and future oriented RF ($\tau \star s = 0$):

$$\tau = \frac{1}{\sqrt{1-v^2}} 1_* - \frac{v}{\sqrt{1-v^2}} j_* \equiv e^{j_* \psi(v)}.$$

$$s = -\frac{v}{\sqrt{1-v^2}} 1_* + \frac{1}{\sqrt{1-v^2}} j_* \equiv j_* e^{j_* \psi(v)}.$$

34

For any event $h = t + jx$:

$$h_T^\tau \equiv \tau(h) = \frac{t - vx}{\sqrt{1 - v^2}},$$

$$h_X^\tau \equiv s(h) = \frac{x - vt}{\sqrt{1 - v^2}}.$$

(Lorentz transformations).

Metric and areas:

$$\epsilon = \tau \otimes \tau - s \otimes s; \quad \text{vol} = \tau \wedge s.$$

For any $\tau_1(v_1)$ and $\tau_2(v_2)$:

$$\tau = \tau_1 \cdot \tau_2 = \frac{1}{\sqrt{1 - v^2}} 1_* - \frac{v}{\sqrt{1 - v^2}} j_*,$$

where

$$v = \frac{v_1 + v_2}{1 + v_1 v_2} \tag{38}$$

— relativistic law of velocities addition (passive viewpoint).

35

16. Conformal relativity theory

$\mathcal{P}(1,1) \to \mathcal{H}\mathrm{ol}(H)$ $(\dim \mathcal{H}\mathrm{ol}(H) = \infty!)$

Locally, for $F \in \mathcal{H}\mathrm{ol}(H)$:

$$(dF)_* : T_h\mathcal{M}_2 \to T_h\mathcal{M}_2, \quad (dF)_*(\xi) = F'\cdot\xi.$$

In exponential representation $F' = I_F|F'|e^{j\psi_F}$, and we obtain the following local ingredients of $\mathcal{H}\mathrm{ol}(H)$:

1. Lorentz transformations, depending on the point $(\psi_F(t,x))$;
2. reflections of ST (I_F);
3. lengths dilations $(|F'|(t,x))$.

New circumstance: variation of norms of 2-vectors!

36

Physical interpreting:
$\mathcal{H}\mathrm{ol}(H)$ **makes transitions between generalized reference frames!**

$$\mathfrak{u} = \frac{dF}{dh} = \frac{\partial U}{\partial t} + j\frac{\partial U}{\partial x}, \tag{39}$$

— 1-form of time with non-unit norm:

$$\|\mathfrak{u}\|^2 = \|\nabla U\|^2 = -\|\nabla V\|^2 = \|F'\|^2. \tag{40}$$

Intervals of proper time $d\tau$ and geometrical lengths ds along world lines are different now!

$$\begin{aligned} \frac{d\tau}{ds} &= \nabla U \star w, \Rightarrow \Delta\tau_{PP'} = \Delta U_{PP'}; \\ \frac{d\ell}{ds} &= \nabla V \star w, \Rightarrow \Delta\ell_{PP'} = \Delta V_{PP'}; \end{aligned} \tag{41}$$

where $w = dh/ds$ ($|w| = 1$).

37

The effect of conformal deformation of clock rate

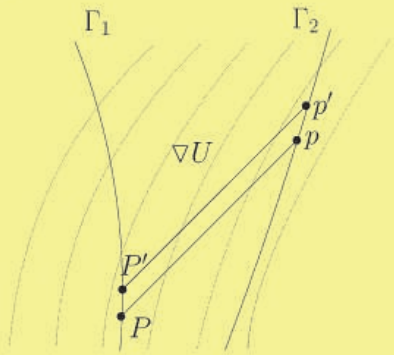


Fig. 25. A pair of clocks in CRT.

Result of calculations in CRT:

$$\delta(P|p) \equiv \frac{d\tau_1}{d\tau_2} = \frac{\Delta_P}{\Delta_p}, \qquad (42)$$

where

$$\Delta_p \equiv \frac{dU/ds_2}{\dot{t} - \dot{x}}; \quad \Delta_P \equiv \left.\frac{dU/ds_1}{\dot{T} - \dot{X}}\right|_{\mathsf{Con}(p)}.$$

38

Example:
weak conformal wave:

$$F(h) = h + \varepsilon \sin \omega h, \quad \omega\varepsilon \ll 1. \qquad (43)$$

$U(t,x) = t + \varepsilon_1 \sin \Phi_1 \cos \Phi_2 + \varepsilon_2 \sin \Phi_2 \cos \Phi_1;$
$V(t,x) = x + \varepsilon_1 \sin \Phi_2 \cos \Phi_1 + \varepsilon_2 \sin \Phi_1 \cos \Phi_2,$
where $\Phi_1 = \omega_1 t + \omega_2 x$, $\Phi_2 = \omega_2 t + \omega_1 x$.

On the pair of spatially remote identical immobile periodic devices (L is distance):

$$\frac{\omega_1}{\omega_2} - 1 = \varepsilon A \sin(\omega t + \varphi_0).$$

— (space-)time modulation of relative frequencies!

(Motivation of experiments of RI HSGP with quartz generators).

39

17. Algebraic theory of space-time-matter

(the "Theory of Everything" in Hyperland)

F — complex potential (describes the whole system — the universe of Hyperland).

$F \in \mathcal{H}\mathrm{ol}(H)$ outside the sources, inside the sources holomorphicity is lost.

Scalars (conformal!) — NH and KT:

$$\mathcal{X} \equiv \|N\|^2, \ (N \equiv \frac{\partial F}{\partial \bar{h}}); \quad \mathcal{Y} \equiv \|K\|^2, \ (K \equiv \frac{\partial F}{\partial h}).$$

Action for F:

$$\mathcal{S}[F, \bar{F}] = \int\limits_H (\mathcal{Y} - \mathcal{U}(\mathcal{X}))\, dh \wedge d\bar{h}, \qquad (44)$$

Field equations:

$$\frac{1}{4}\Box F = (\mathcal{U}' F_{,\bar{h}})_{,h} \qquad (45)$$

— non-linear wave equation (hyperbolic version of Mie theory).

40

The first integral:

$$F_{,\bar{h}}(1-\mathcal{U}') = \varphi(\bar{h}), \qquad (46)$$

$\varphi(\bar{h})$ is an arbitrary antiholomorphic function.

Consequence:

$$\Box \text{Arcth}\left[\frac{N_1}{N_2}\right] = 0 \Rightarrow N_1 = N_2(\phi^+ + \phi^-)$$

— boundaries of sources in Hyperland are lines (rather than points).

EMT $\mathcal{T}$ components in complex basis:

$$\mathcal{T}^h_h = \mathcal{T}^{\bar{h}}_{\bar{h}} = \mathcal{U}(\mathcal{X}) - \mathcal{U}'(\mathcal{X})\mathcal{X} \equiv \mu;$$

$$\mathcal{T}^h_{\bar{h}} = \overline{\mathcal{T}^{\bar{h}}_h} = \bar{F}_{,\bar{h}}F_{,\bar{h}}(1-\mathcal{U}') \equiv \sigma.$$

Energy density and pressure:

$$\varepsilon = \mu + \sqrt{\sigma\bar{\sigma}} = \mathcal{U} - \mathcal{U}'\mathcal{X} + (1-\mathcal{U}')\sqrt{\mathcal{X}\mathcal{Y}};$$

$$p = \mu - \sqrt{\sigma\bar{\sigma}} = \mathcal{U} - \mathcal{U}'\mathcal{X} - (1-\mathcal{U}')\sqrt{\mathcal{X}\mathcal{Y}}.$$

41

Functional determinant

$$\frac{D(\varepsilon,p)}{D(X,Y)} \equiv \begin{vmatrix} \varepsilon_{,X} & \varepsilon_{,Y} \\ p_{,X} & p_{,Y} \end{vmatrix} = \frac{\mathcal{U}''(1-\mathcal{U}')\mathcal{X}^2}{\mathcal{Y}^{1/2}} \neq 0$$

— ε and p are not connected to each other by some equation of state $p = f(\varepsilon)$.

Where does $\mathcal{U}$ come from?

42

18. Extravariational principle

Let $\mathcal{S}_\alpha[\phi]$ be the action for some fundamental theory (α — the set of constants) and let $\phi_\beta(\alpha)$ be the solution of the Euler-Lagrange equations:

$$\delta_\phi \mathcal{S}_\alpha[\phi] = 0,$$

where β are initial-boundary conditions parameters. Substituting the solution back into the action (after suitable regularization if necessary), we obtain the function:

$$\Phi(\alpha, \beta) \equiv \mathcal{S}_{\text{reg}}\ \alpha[\phi_\beta(\alpha)]. \qquad (47)$$

The key idea of the extravariational procedure (for the constants):

$$\Phi(\alpha, \beta) \to \min_{\alpha,\beta}$$

— calculation of fundamental constants "from nothing" !

43

Example: harmonic oscillator

$$\mathcal{S}[x(t)] = \int_0^T \left[\frac{m\dot{x}^2}{2} - \frac{kx^2}{2}\right] dt \overset{x=A\sin(\omega t+\varphi)}{=}$$

$$\cdots = \bar{E}T \cdot \frac{\sin 2\chi - \sin 2\varphi}{\chi - \varphi}, \quad (48)$$

$\chi = \omega T + \varphi.$

The extremum condition:

$$\chi = \varphi + \pi n \Rightarrow \omega = \omega_n = \frac{\pi n}{T}$$

— relation between constant + "quantization".

44

Extravariational procedure for potential function

$$\mathcal{S}[\phi] = \int \mathcal{L}(\phi, \partial\phi) d\text{vol}, \qquad (49)$$

— action of some field theory with Lagrangian $\mathcal{L} = (\partial\phi)^2 - U(\phi, \partial\phi)$.

The system of integrals:

$$F(\phi, \partial\phi) = 0 \overset{?}{\Rightarrow} \mathcal{L} \to \mathcal{L}'(\phi, U, \partial U) \qquad (50)$$

Going to the new coordinates $\{x\} \to \{x', y\}$, where $y = \phi$:

$$\mathcal{S}[\phi]|_F = \int \mathcal{L}'(y, U, \partial U) J|_F \, dx' \wedge dy \equiv \mathcal{S}'[U(y)]$$

— functional with respect to $\mathcal{U}(y)$.

$$\delta_U \mathcal{S}'[U(y)] = 0, \qquad (51)$$

— the equation for U.

45

Extravariational principle in Hyperland

Using the integral:

$$\mathcal{Y} \overset{\text{up to b. t.}}{=} \mathcal{X} = \frac{\|\varphi\|^2}{(1-\mathcal{U}')^2}, \qquad (52)$$

we go to the functional:

$$\mathcal{S}'[\mathcal{U}] = \int \left[\frac{\|\varphi\|^2}{(1-\mathcal{U}')^2} - \mathcal{U} \right] d\mathcal{X} \wedge d\mathcal{X}'.$$

Equation of the extremal:

$$\frac{d}{d\mathcal{X}} \left(\frac{\mathcal{X}}{1-\mathcal{U}'} \right) = -\frac{1}{2}.$$

<u>Solution</u>: $\mathcal{U}^{\checkmark}(\mathcal{X}) = 3\mathcal{X} + \mathcal{U}_0 - 2\mathcal{U}_1 \ln \left| 1 + \frac{\mathcal{X}}{\mathcal{U}_1} \right|$, where $\mathcal{U}_0, \mathcal{U}_1$ is a pair of "fundamental constants" of Hyperland.

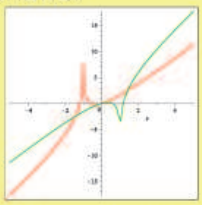


Fig. 26. $U^{\checkmark}(\mathcal{X})$ under $\mathcal{U}_0 = 0, \mathcal{U}_1 = \pm 1$

46

Static Hyperland

Non-holomorphicity:

$$\mathcal{X} = \frac{1}{4}(V_{,x}^2 - U_{,x}^2) = -\frac{U_{,x}^2}{4\cosh^2(Ax)}.$$

Substituting $\mathcal{X}$ in the action with $\mathcal{U}^{\checkmark}$ and varying it on $U(x)$, we obtain after integration:

$$Z^3 + \alpha\cosh(Ax)(Z^2 - \epsilon) = 0 \qquad (53)$$

where

$$Z = \frac{U_{,x}}{2\sqrt{|\mathcal{U}_1|}\cosh(Ax)}, \ \epsilon \equiv \text{sign}(\mathcal{U}_1).$$

The action:

$$\mathcal{S}_0 = \mathcal{U}_1 T \int\limits_{-L/2}^{L/2} \left(Z^2 + \epsilon\ln\left(1 - \epsilon Z^2\right) - \Lambda\right) dx,$$

$$\Lambda = \mathcal{U}_0/\mathcal{U}_1.$$

47

Fundamental constants of static Hyperland

K	Meaning	Value
Λ	add. norm. of energy	**important,=?**
$\mathcal{U}_1$	mult. norm of energy	unimportant
A	length scale	unimportant
α^2	dim-less "interact. const."	**important, =?**
L	scales	∞
T	of ST	∞

The universe of static Hyperland is determined by a single "fundamental constant" α^2!

48

Calculation of α

From the equation (53) we obtain:

$$\frac{\partial Z}{\partial \alpha} \sim Z; \ \frac{\partial Z}{\partial x} \sim \frac{\sqrt{Z^6 - \alpha^2(\epsilon - Z^2)^2}(\epsilon - Z^2)}{Z^2(3\epsilon - Z^2)}.$$

The condition $\partial \mathcal{S}_0/\partial \alpha = 0$:

$$\int\limits_{s^*(\alpha^2)}^{\infty} \frac{s^{5/2}(3 - \epsilon s)}{(1 - \epsilon s)^2 \sqrt{s^3 - \alpha^2(1 - \epsilon s)^2}} ds = 0, \tag{54}$$

where $s^{*3} - \alpha^2(1 - \epsilon s^*)^2 = 0$, $s = Z^2$.

The integral (54) is divergent ($O(s)$ for $s \to \infty$ and poles at finite points)!

Only for $\epsilon = +1$ there exists a unique value α^2, which removes the finite pole:

$$\alpha^2 = \alpha_0^2 = \frac{27}{4}. \tag{55}$$

49

Energy density and pressure of the matter (dublon) of Hyperland:

$$\left\{ \begin{matrix} \varepsilon \\ p \end{matrix} \right\} = \Lambda - 2\ln|1 - Z^2| - \frac{2Z^2(1 \pm Z^2)}{1 - Z^2}.$$

Coordinate and length:

$$\cosh x = \frac{Z^3}{\alpha_0(1 - Z^2)}; \; r = V = \frac{Z^4 - Z^2 + 2}{1 - Z^2} + \ln|1 - Z^2|.$$

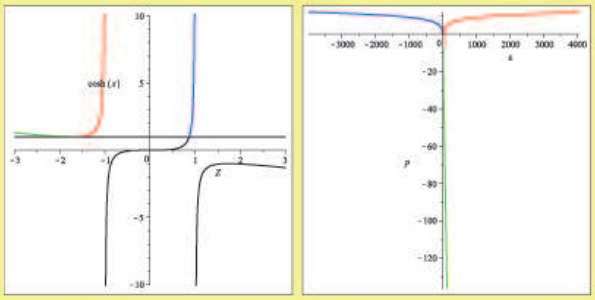


Fig. 27. Three physical regions of Z and three branches of state equation of dublon — blue, red and green states of dublon.

50

"Blue dublon".

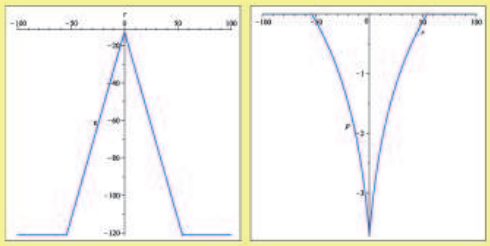


Fig. 28. $\varepsilon(r)$ and $p(r)$ of the "blue dublon" $(\Lambda = -5)$.

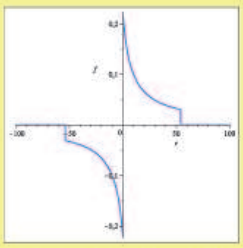


Fig. 29. $f(r)$ inside the "blue dublon"

Fundamental repulsivity inside the "blue dublon":

$$f \overset{r\to\infty}{=} \frac{2}{r}; \quad f \overset{r\to 0}{=} \frac{2}{9}.$$

51

"Red dublon".

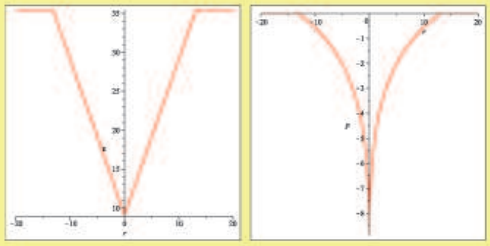


Fig. 30. $\varepsilon(r)$ and $p(r)$ of the "red dublon" $(\Lambda = -1.5)$

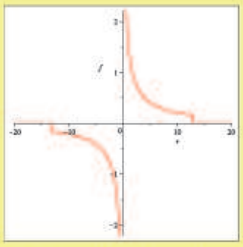


Fig. 31. $f(r)$ inside the "red dublon."

Fundamental repulsivity inside the "red dublon":

$$f \overset{r\to\infty}{=} \frac{2}{r}; \quad f \overset{r\to 0}{=} \frac{\sqrt{6}}{\sqrt{r}}.$$

52

"Green dublon".

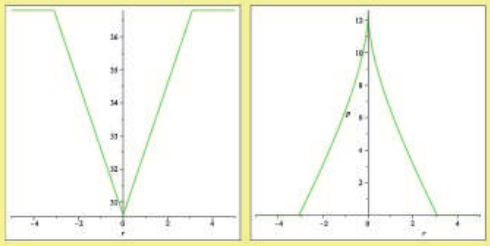


Fig. 32. $\varepsilon(r)$ and $p(r)$ for the "green dublon" $(\Lambda = 20)$.

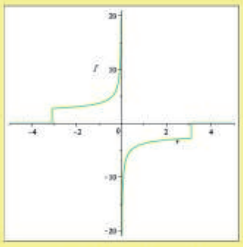


Fig. 33. $f(r)$ inside the "green dublon"

There is fundamental attraction inside the green dublon:

$$f \overset{r\to\infty}{=} 2 - \frac{4}{r}, \quad f \overset{r\to 0}{=} -\frac{\sqrt{6}}{\sqrt{r}}.$$

53

19. What next?

$$H = P_2 \subset P_3 \subset \ldots P_n \subset \ldots$$

— hierarchy of commutative-associative algebras of polynumbers.

$$A = A_1 e_1 + \ldots A_n e_n, \quad e_i e_j = e_j \delta_{ij}.$$

$$\epsilon = {}^{(2)}\epsilon \leftarrow {}^{(3)}\epsilon \leftarrow \cdots \leftarrow {}^{(n)}\epsilon \leftarrow \ldots$$

— hierarchy of (Finslerian!) Berwald-Moor spaces:

$${}^{(n)}\epsilon = \hat{S}(dx^1 \otimes \cdots \otimes dx^n).$$

Scalar polynumber operator (Garas'ko operator):

$$\bigcirc n \equiv \frac{\partial^n}{\partial \overset{1}{x} \ldots \partial \overset{n}{x}}$$

Spherically-symmetric fundamental solution ($r = \|x\|^{1/n}$):

$$R_n(r) = C_0 + C_1 \ln r + \cdots + C_{n-1} \ln^{n-1} r.$$

54

Co-tangent construction

(key to physical interpretation of BM metric)

$V(H_4)\times V(H_4):\quad {}^{(4)}\epsilon \to g \equiv {}^{(4)}\epsilon(U,V,\ ,\ )$

or in index notation: $g_{\alpha\beta} = {}^{(4)}\epsilon_{\alpha\beta\gamma\delta}U^{\gamma}V^{\delta}$.

Locally (at fixed point) there are only two non-degenerated types: $(+,-,-,-)$ and $(+,+,-,-)$ (vector spaces, connected with complex and hyperbolic spinors!).

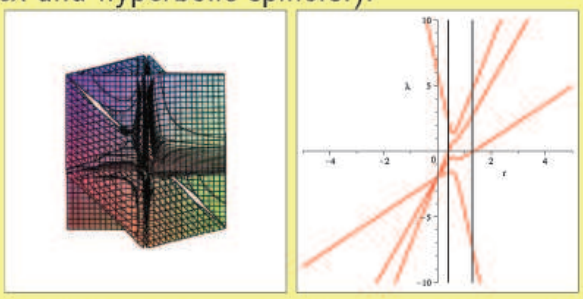


Fig. 34. Projection of surfaces of degeneration of co-tangent metric and behaviour of its signature along the line $(1-t, 1-2t, 1-3t, 1-0.5t)$

55

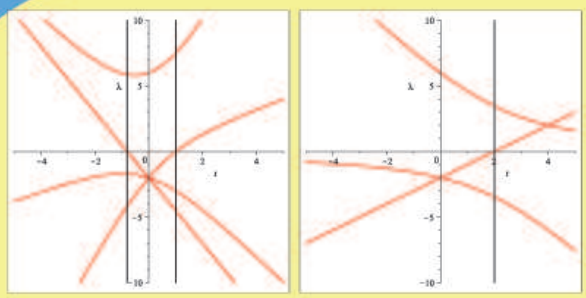


Fig. 35. Behaviour of signature along the lines $(1-t, 1+2t, 1-t, 1+0.5t)$ and $(1-t, 1, 1, 1)$

Co-tangent Riemannian metric can't be Euclidean $(\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 = 0)$**!**

56

20. Appendix

1. **Geometrization principle.**
   GR and Weyl theory, gauge fields, superstring theory.

$$A = \nabla_v v = 0 \tag{56}$$

   — universal equation of motion. Contemporary geometry and physical reality. Criteria of selection.

2. **Algebraization principle.**
   Quantum theory, symmetry, Lie groups and algebras, TFCV and its applications, p-adic analysis, category and toposes theory. Simbiosis of algebra and geometry.

57

3. **Poincare's geometric conventionalism.**
The nature of geometry and its role in Physics. The question on the "natural geometry" of the world. Interplay of geometry and physics:

Geom. + Ph.Pr. = Ph.Th. ⇔ Exp.

4. **Generalized and alternative theories.**

Characteristics	Generalized	Alternative
Motivation	Expansion	Reconsideration
Construction	Deformation	Rejection
Succession principle	Limit procedure	Morphisms dictionary
The role of contemp. experiments	Crucial	Indirect

58

Examples of alternatives:

Faraday-Maxwell electrodynamics
and
Wheeler-Feynman electrodynamics;

GR, QM
and
twistor physics of R. Penrose;

SM and BGP of Yu.S.Vladimirov;

.....

History and methodology of science, psychological factors. Non-phenomenological nature of future "Theory of Everything."

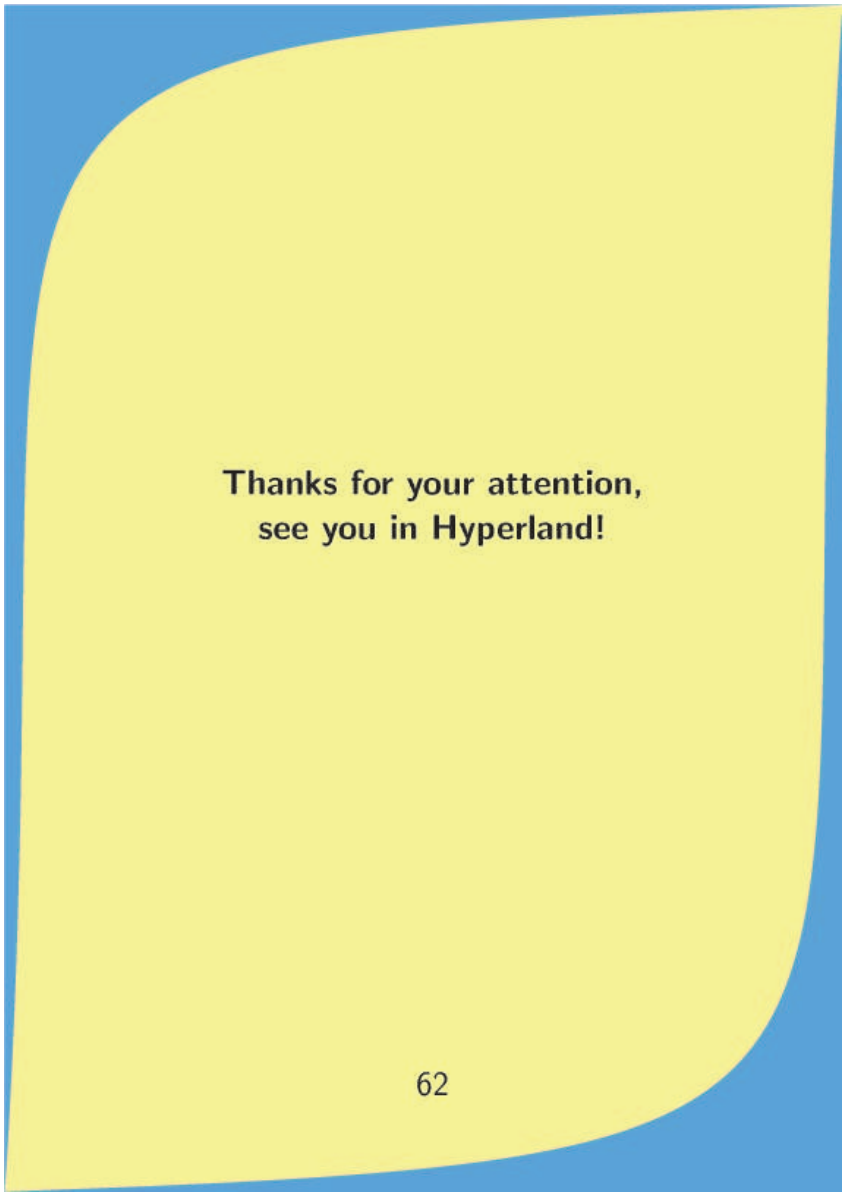
59

References

1. R. Penrose, *The Road to Reality (A Complete Guide to the Laws of the Universe)*, M.-Izhevsk, 2007.
2. R.Penrose, W. Rindler, *Spinors and space-time (v.1,2)*, M., Mir, 1987.
3. V. Balan, G.Yu. Bogoslovskiy, S. S. Kokarev, D. G. Pavlov, S.V. Siparov, N. Voicu, *Geometrical Models of the Locally Anisotropic Space-Time*, Journal of Modern Physics, 2012, 3, pp.1314-1335 (arXiv:1111.4346)
4. G.I. Garas'ko, *Fundamentals of Finslerian geometry for physicists*, M.: Tetru, 2009 (In Russian)
5. D.G.Pavlov, S.S.Kokarev, *h-holomorphic functions and its applications*, HNGP 1(13), v.7 2010, p.44-77 (In Russian)
6. D.G.Pavlov, S.S.Kokarev, *Hyperbolic field theory on the plane of double variable*,

HNGP 1(13), v.7, 2010, p.78-126 (In Russian)

7. D.G.Pavlov, S.S.Kokarev, *Algebraic unified theory of space-time-matter on the plane of double variable,* HNGP 2(14), v.7, 2010, p.11-37 (In Russian)
8. S.S.Kokarev, *Extravariational principle in the theory of field,* In coll. papers of RSEC "Logos" 6, 2011, pp. 123-146 (In Russian)
9. S.S.Kokarev, *Three lectures on Newton's law,* In coll. papers of RSEC "Logos" 1, 2006, pp.45-72 (arXiv:0905.3285v1[gr-qc]).
10. D.G.Pavlov, M.S.Pancheluga, V.A.Pancheluga, *To the form of the Julia set analog on the plane of double variable,* HNGP, (2)12, 6 (2009), pp.161-175 (In Russian)
11. D.G.Pavlov, M.S.Pancheluga, V.A.Pancheluga, *Exploring researches on the space-time effects of hyperbolic fields. Preliminary results,* HNGP, (1)172, 9 (2012), pp.162-175.


Thanks for your attention,
see you in Hyperland!
62